Maximum of a Stationary Random Field

K F Turkman,
CEAUL-FCUL, University of Lisbon

Research partially funded by FCT Fundação para a Ciência e a Tecnologia through the projects PEst-OE/MAT/UI0006/2011 and PTDC/MAT/118335/2010
\[ \{X_i(s)\} \text{ is an iid replicate of an random field } X(s) \text{ and } \]
\[ a_n(s), b_n(s) \text{ are suitably chosen normalising functions.} \]

\[ \lim_{n \to \infty} \frac{1}{a_n(s)}(\max_{1 \leq i \leq n} X_i(s) - b_n(s)) \to Y(s). \]

A max-stable process \( Y(s), s = (s_1, s_2) \in S \subset \mathbb{R}^2. \)

Davison and Gholamrezaee(2012) for applications. de Haan and Ferreira(2006) for the structure of the possible limit processes \( Y(s) \) and the characterization of the domains of attraction.
Max-stable processes provide a natural extension of multivariate extremes to process setting and have found many practical applications in modeling, simulating and understanding of clustering of rare events generated by the random field $X(s)$ in space.

We will be looking at different limiting operations and limiting results.
One particular quantity of interest is approximations for the excursion probability

\[ P(\sup_{s \in S} X(s) \geq u), \]

for some d-dimensional indexing set \( S \) with finite volume \( V(S) \) or when the volume \( V(S) \) increases in an controlled fashion.

\( Z(s) = \sup_{s \in S} X(s) \) and the max-stable limit \( Y(s) \) are not related.
Excursion probabilities II

- $Y(s)$ appears as limit for a large class of initial random fields $X(s)$ whereas $Z(s)$ is a functional of the original process and one can not infer to $Z(s)$ based on the max-stable limiting operation.
- Therefore, Max-stable processes that appear as the limit are not particularly useful in estimating such excursion probabilities and different, alternative models and approximations are needed to estimate such probabilities.
- A limiting operation known as the Pickand’s double sum method gives rise to well known asymptotic methods and models for the excursion probability in question.
<table>
<thead>
<tr>
<th></th>
<th>Pickand’s double sum method basically works in the following fashion:</th>
</tr>
</thead>
<tbody>
<tr>
<td>□</td>
<td>Start with a sufficiently fine grid (lattice) over $S$,</td>
</tr>
<tr>
<td>□</td>
<td>Characterize the extremal properties of the process $X(s)$ over this fine grid,</td>
</tr>
<tr>
<td>□</td>
<td>Then obtain asymptotical results by letting the grid sizes go to 0.</td>
</tr>
<tr>
<td>□</td>
<td>Bickel and Rosenblatt (1973) for the first use of this method in obtaining limiting results for Gaussian random fields,</td>
</tr>
</tbody>
</table>
Difficulties

- There is a large body of literature concerned with the extremal properties of discrete and continuous parameter stochastic processes defined over one dimensional index set (i.e., time) based on this limiting operation.
- Surprisingly few literature on the extremal properties of random fields based on this limiting operation, except for Gaussian random fields.
- Difficult to extend basic notions such as the number of up-crossings to random fields, simple ordering (in time) often no applicable for random fields.
- Piterbarg (1996), Sun (1993) and French and Davis (2010) for different approximations and characterizations of the excursion probability for Gaussian Random fields.
Objective 1

- Give an asymptotic expression for

\[ P(\sup_{s \in S} X(s) > u), \]

as \( u \to \infty \) for fixed intervals \( s \in S = [0, h_1] \times [0, h_2] \), and for increasing \( h_1, h_2 \).

- In terms of the stationary distribution

\[ P(X(0,0) > x) = F(x), \]

as well as in terms of the local clustering of large values of the process along the spatial coordinates.


- \( F(x) \) belongs to the Frechét domain of attraction, however extension to other domains of attraction is possible.
Objective II

- O’Brien type results for sequences of random variables:
  \[ P(M_n \leq u_n) - [P(X_1 \leq u_n)]^{nP(M_1,p_n|X_1>u_n)} \rightarrow 0 \]

- Characterizing the limiting distribution of the maximum in terms of local clusters

- Objective: obtain similar characterizations for the maxima of a random field.
Some definitions

- $X(s_1, s_2)$ be a stationary but not necessarily isotropic random field.
- $X(0, s)$ Column-wise process at $x$-coordinate $s_1 = 0$
- $Y(s) = \max_{0 \leq s_2 \leq h_2} X(s, s_2)$, $s \in [0, h_1]$ of column maxima process at each $x-$coordinate location $s$.
- $Y(0) = \max_{0 \leq s \leq h_2} X(0, s)$.
- We assume that the maximal process $Y(s)$ as well as the process $X(0, s)$ satisfy the Albin type clustering conditions, characterizing the clustering of extreme events of the random field $X(s_1, s_2)$ in terms of the clustering of extreme events of the coordinate-wise processes $Y(s_1)$ and $X(0, s_2)$. 
Albin Conditions for $X(0,s)$

- $F$ belongs to the Frechet domain of attraction.
- **Clusters near a local maximum:** There exists random variables $\{\zeta_a(k)\}_{k=1}^{\infty}$ and a positive (nonincreasing) function $q = q(u)$ with $\lim_{u \to \infty} q(u) = 0$, such that

$$\left(\frac{1}{u}X(0, aq), \ldots, \frac{1}{u}X(0, aqN_2)\right|\frac{1}{u}X(0,0) > x\right) \to^D \{\zeta_{a,x}(k)\}_{k=1}^{N_2}$$

for all fixed integers $N_2$ and $x > 0$.

- **Minimal discrete approximation along the $y$-direction**

$$\lim_{u \to \infty} \frac{q(u)}{1 - F(u)} P(M(0,h_2) > u, \max_{0 \leq aqk \leq h_2} X(0,aqk) \leq u) = 0,$$

when $a \to 0$. 


limits for $X(0, s)$

- **Short-lasting-exceedances along $y$-direction**

- The limit

  $$\lim_{a \to 0} \frac{1}{a} P(\max_{k \geq 1} \zeta_{a,x}(k) \leq x) = H_2(x) \in (0, \infty)$$

  exists and

  $$\lim_{u \to \infty} \frac{q(u)}{1 - F(ux)} P(Y(0) > ux) = h_2 H_2(x).$$

- **Note:** $Y(0) = M(0, h_2) = \max_{0 \leq s \leq h_2} X(0, s)$. 
limits for column-max process $Y(s)$

- We will assume a similar set of conditions on $Y(s)$: Let $P(Y(0) \leq u) = G(u)$.
- There exists variables $\{\eta_b(k)\}_{k=1}^{\infty}$ and a strictly positive function $g = g(u)$ with $\lim_{u \to \infty} g(u) = 0$, such that

$$\left(\frac{1}{u}Y(bg), \ldots, \frac{1}{u}Y(bgN_1)|\frac{1}{u}Y(0) > x\right) \to D \{\eta_{b,x}(k)\}_{k=1}^{N_1},$$

for every fixed integer $N_1$.

- **Short lasting exceedance along the $x$-direction**
- **Discrete approximation to $Y(s)$**: 

$$\lim_{u \to \infty} \frac{g(u)}{1 - G(u)} P(M_Y(h_1) > u, \max_{0 \leq bgk \leq h_1} Y(bgk) \leq u) = 0,$$

when $b \to 0$. 
limits for $Y(s)$

\[ \lim_{b \to 0} \frac{1}{b} P(\max_{k \geq 1} \eta_{b,x}(k) \leq x) = H_1(x) \in (0, \infty) \]

\[ \lim_{u \to \infty} \frac{g(u)}{1 - G(ux)} P(M_Y(h_1) > ux) = h_1H_1(x) \]

\{ (aq(u)i, bg(u)j), i = 0, 1, 2, \ldots, j = 0, 1, \ldots \},

Pickand grid; the minimal grid needed for discrete approximations with sufficient accuracy,
Under the conditions given for column-wise processes, 

\[
\lim_{u \to \infty} \frac{q(u)g(u)P(M(h_1, h_2) > ux)}{(1 - F(ux))} = h_1 h_2 H_1(x) H_2(x),
\]

\[
\lim_{a \to 0} \frac{1}{a} P(\max_{k \geq 1} \zeta_{a, x}(k) \leq x) = H_2(x) \in (0, \infty)
\]

\[
\lim_{b \to 0} \frac{1}{b} P(\max_{k \geq 1} \eta_{b, x}(k) \leq x) = H_1(x) \in (0, \infty)
\]
Heuristically, the proof of this theorem follows from Albin (1987, 1990),

\[ \lim_{u \to \infty} \frac{g(u)P(M(h_1, h_2) > ux)}{P(M(0, h_2) > ux)} = h_1 H_1(x), \]

\[ \lim_{u \to \infty} \frac{q(u)P(M(0, h_2) > ux)}{1 - F(ux)} = h_2 H_2(x), \]

Rigorous proof follows from extension of O’Brien (1987) techniques.
Maxima over increasing intervals

- $S = [0, h_1] \times [0, h_2]$, and
  \[
  M(S) = \max_{(s_1, s_2) \in S} X(s_1, s_2).
  \]

- We now look at
  \[
  P(M(S) > u_h x),
  \]
  as $h = h_1 h_2 \to \infty$,

- The normalizing constant $u_h$:
  \[
  \frac{h_1 h_2}{g(u_h) q(u_h)} P(X(0, 0) > u_h x) = \tau(x),
  \]
  for some non-degenerate $\tau(x)$,

- $h_1, h_2$ both tend to infinity, although not necessarily with the same rate.
The Cw-mixing condition of Leadbetter and Rootzen (1998) gives us the extremal types theorem needed for characterizing the limiting distribution of $M(S)$:

- $m_1, m_2$ blocks in each $x$ and $y$ directions, each having size $r_1, r_2, h_1 = r_1m_1, h_2 = m_2r_2$.
- $P(M(S) \leq u_h x) = P^{m_1m_2}(M(J) \leq u_h x) + o_h(1)$,
- as $h_1, h_2$, tend to $\infty$, where

$$J = B_{11} = ([0, r_1] \times [0, r_2]).$$

Approximate $P(M(J) \leq u_h x)$ by giving domains of attraction criteria in terms of coordinate-wise clustering of large events by $X(0, s_1)$ and $Y(s)$. 
For each $0 \leq s \leq r_1$, again, let

$$Y(s) = \max_{0 \leq s_2 \leq r_2} X(s, s_2).$$

be the coordinate-wise maximal process at each $x$-direction location $s$.

Local dependence structure of the stationary spatio-temporal process will be characterized by two conditions which represent the propensity of the large values of the process to cluster along $x$ and $y$ coordinates.
coordinate wise clustering

- **x-coordinate clustering propensity**

\[
\lim_{h \to \infty} \frac{1}{aq} P(X(0, aq) \leq uhx, \ldots, X(0, r_2aq) \leq uhx | X(0, 0) > uhx) = \theta_1(x),
\]

for some \(0 < \theta_1(x) \leq 1\), as \(a \to 0\).

- How the large values of the \(X(0, s)\) process clusters along the \(y\) direction at a fixed \(x\)-coordinate.

- **y-coordinate clustering propensity**

\[
\lim_{h \to \infty} \frac{1}{bg} P(Y(bg) \leq uhx, \ldots, Y(r_1bg) \leq uhx | Y(0) > uhx) = \theta_2(x),
\]

for some \(0 < \theta_2(x) \leq 1\), as \(b \to 0\).

- How largest values in the \(y\) columns cluster along the \(x\)-direction.
Theorem 2

\[
\lim_{h \to \infty} |P(M(S) \leq u_h x) - \exp(-\theta_1(x)\theta_2(x)\tau(x))| = 0.
\]

Coordinate-wise clustering conditions are weaker than the Albin conditions.

Under the *Short lasting exceedances and no clusters of clusters* conditions for the processes \(X(0, s)\) and \(Y(s)\), \(\theta_1(x)\) and \(\theta_2(x)\) exist and are equal to \(H_1(x), H_2(x)\)
References